reduced, but the break-up of the jet occurs further from the orifice, and for the lowest pressure not until after the lip of the oscillator or pipe has been passed. In this case, as was before mentioned, only a small amplitude of vibration is maintained, and the effect of the dynamic stiffness of the jet preponderates.

Thus the fact that low wind or water pressure depresses the pitch, while higher pressures raise it, is accounted for, although it is impossible to calculate the amount from purely dynamical data (viz., velocity, pressure, density, and viscosity), as no theory is at present capable of dealing with the formation of eddies or the final result of fluid instability.

The Periods of Lateral Vibration of Loaded Shafts.—The Rational Derivation of Dunkerley's Empirical Rule for Determining Whirling Speeds.

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(Communicated by Prof. W. McF. Orr, F.R.S. Received June 18, 1918.)

The calculation of the whirling speeds of loaded shafts of turbines and turbine-driven machinery is usually rendered complex by reason of the varying distribution of the load along the length of the shaft, as well as by variations from place to place in the shaft's diameter.

Of the several methods in use for the estimation of the first whirling speed in such cases, probably the best known is that based on the Dunkerley hypothesis. This method, which was put forward by Dunkerley in 1893* and illustrated by him in many applications, consists in dividing the load into a convenient number of parts acting at appropriate positions along the shaft, and determining by means of simple formulæ the whirling speed for each such part of the mass regarded as acting alone. It is then assumed that the sum of the squares of the reciprocals of these partial whirling speeds is equal to the square of the reciprocal of the first whirling speed for the whole load.

Expressed algebraically, if ω_1, ω_2, ω_3, etc., be the whirling speeds of the shaft loaded in turn with the separate masses regarded as acting alone, and

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* 'Phil. Trans.,' A, vol. 185, pp. 279–360 (1894), especially §§ 28, 61, 62.

Ω be the first whirling speed of the loaded shaft, then Dunkerley's empirical rule consists in the assumption that

$$\frac{1}{\Omega^2}=\frac{1}{\omega_1{}^2}+\frac{1}{\omega_2{}^2}+\frac{1}{\omega_3{}^2}+\dots$$

In many cases this method gives excellent results, and, having regard to the uncertainty of the data and the purposes for which the calculation is required, quite sufficient accuracy is often obtained. But the method is not invariably safe, and in certain applications may give results very seriously in error.

That the empirical rule is subject to such limitations was known to Dunkerley, who, in the paper referred to, proposed, in certain cases in which the calculated result did not agree with experiment, to substitute a formula of the type (for two loads)

$$\frac{1}{\Omega^2}=\frac{1}{\omega_1{}^2}+\frac{a}{\omega_2{}^2},$$

where a has a value less than unity.[*]

The need for such modification in certain cases was also pointed out by Chree.[†] It is not, however, clear beforehand what cases require modification, or the extent of the modification, *i.e.* the value to give a.

It is proposed in the following note to show how Dunkerley's empirical rule may be derived rationally, and as a consequence to show its limitations, and how it may be extended to give more accurate results in any application.

It is to be noted that the method used below is of general application, but to fix the ideas and render the argument clearer, the case will be taken of a uniform and light shaft freely supported by bearings at its ends, and loaded with a number of discrete masses $m_1, m_2, \dots, m_n$, of negligible moment of inertia at definite points along its length.

As above, let $\omega_1, \omega_2, \dots, \omega_n$, be the whirling speeds corresponding to each of the masses $m_1, m_2, \dots, m_n$, acting alone; and let $\Omega_1, \Omega_2, \dots, \Omega_n$, be the first, second, etc., whirling speeds of the loaded shaft with all loads in position, corresponding to the different possible types of vibration. In any assumed possible type of lateral bending of the shaft under the action of centrifugal forces and subject to the end restraints, let $u_1, u_2, \dots, u_n$, be the lateral displacements from the straight form at the points where the masses $m_1, m_2, \dots, m_n$, are fixed.

In the first instance, consider the shaft subjected simply to a force X_1 acting at right angles to its length l at a point distant z_1 from one end. Let

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[*] Dunkerley, *ibid.*, §§ 30, 33, 44.

[†] 'Phil. Mag.,' May, 1904, p. 541.

E be Young's modulus for the shaft and I its moment of inertia of cross-section about a diameter. Then, following the Bernoulli-Eulerian method, since u is the displacement at the point z parallel to X, we have, for the shaft from $z = 0$ to $z = z_1$,

$$-\mathrm{EI}\frac{d^2u}{dz^2} = \mathrm{X}'z,$$

where X′ is the force exerted at the bearing at $z = 0$.

Similarly for the portion of the shaft z_1 to l we have

$$-\mathrm{EI}\frac{d^2u'}{dz^2} = \mathrm{X}''(l-z),$$

where X″ is the force exerted at the bearing at $z = l$.

Integrating we obtain

$$6\mathrm{EI}u = 6\mathrm{EI}u_1\frac{z}{z_1} + \mathrm{X}'z(z_1^2 - z^2)$$

and

$$6\mathrm{EI}u' = 6\mathrm{EI}u_1\frac{l-z}{l-z_1} + \mathrm{X}''(l-z)(z-z_1)(2l-$$

where u_1 is the value of u at z_1. Now, neglecting "th
shearing," the curves have a common tangent at z_1. Henc

$$3\mathrm{EI}lu_1 = \mathrm{X}'z_1^3(l-z_1) + \mathrm{X}''z_1(l-z_1)^3.$$

But

$$\mathrm{X}' = \mathrm{X}_1\frac{l-z_1}{l}, \quad \text{and} \quad \mathrm{X}'' = \mathrm{X}_1\frac{z_1}{l}.$$

Hence

$$6\mathrm{EI}lu = \mathrm{X}_1 z(l-z_1)(2lz_1 - z_1^2 - z^2),$$

and

$$6\mathrm{EI}lu' = \mathrm{X}_1 z_1(l-z)(2lz - z^2 - z_1^2).$$

These equations give the displacement at any point along the shaft consequent on the action of the force X_1 at z_1.

If now a number (n) of such forces X act at several points along the shaft, the deflection at any point will be the sum of the separate deflections at that point due to each of the forces acting separately. Accordingly we obtain

$$6\mathrm{EI}lu_r = \sum_{s=0}^{s=r}\mathrm{X}_s z_s(l-z_r)(2lz_r - z_r^2 - z_s^2) + \sum_{s=r+1}^{s=n}\mathrm{X}_s z_r(l-z_s)(2lz_s - z_s^2 - z_r^2),$$

where the first summation includes all the forces on the shaft from 0 to z_r, and the second summation includes all those from z_r to l.

If, now, the forces X_s, etc., be due to the centrifugal action of the masses m_s, etc., when the shaft is rotating with uniform angular velocity Ω in a deflected form, u_s, etc., being the corresponding displacements, then $\mathrm{X}_s = m_s\Omega^2u_s$, etc.

If the shaft is not rotating, when determining the speed of lateral vibration we put $\mathrm{X}_s = m_s\Omega^2u_s$ in the same way, where Ω is now the speed of lateral vibration.

Substituting we obtain

$$6\mathrm{EI}lu_r = \Omega^2 \sum_{s=0}^{s=r} m_s z_s (l-z_r)(2lz_r - z_r^2 - z_s^2)\, u_s$$
$$+\Omega^2 \sum_{s=r+1}^{s=n} m_s z_r (l-z_s)(2lz_s - z_s^2 - z_r^2)\, u_s,$$

where the first summation extends to each of the masses up to and including m_r towards one end of the shaft, and the second summation to those masses beyond m_r towards the other end.

Putting $\frac{6\mathrm{EI}l}{\Omega^2} = \Phi$, the above equation may be written in the more convenient form

$$\Phi u_r = f_1 u_1 + f_2 u_2 + \ldots + f_n u_n,$$

where the coefficients f_1, f_2, etc., are functions of the magnitudes and positions of the masses m_1, m_2, etc., and of z_r.

Taking m_r to be each of the masses $m_1, m_2, \ldots, m_n$, in turn, we have the equations

$$\Phi u_1 = a_1 u_1 + a_2 u_2 + \ldots + a_n u_n,$$
$$\Phi u_2 = b_1 u_1 + b_2 u_2 + \ldots + b_n u_n,$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$\Phi u_n = h_1 u_1 + h_2 u_2 + \ldots + h_n u_n.$$

It is worthy of note that the coefficients are related to one another in a symmetrical manner, viz.

$$\frac{a_2}{b_1} = \frac{m_2}{m_1}, \quad \frac{a_3}{c_1} = \frac{m_3}{m_1}, \quad \frac{b_3}{c_2} = \frac{m_3}{m_2}, \text{ etc.,}$$

a result to be expected from the well known theorem that the displacement at a point A of a beam due to a load at another point B, is equal to the displacement at B due to the same load acting at A.*

Eliminating $u_1, u_2, \ldots, u_n$, we obtain the determinant

$$\begin{vmatrix} a_1 - \Phi, & a_2, & \ldots & a_n \\ b_1, & b_2 - \Phi, & \ldots & b_n \\ \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots \end{vmatrix} = 0.$$

The roots of this equation in Φ give the n whirling speeds of the shaft loaded with the n masses $m_1, m_2, \ldots, m_n$, all on together, viz., $\Omega_1, \Omega_2, \ldots, \Omega_n$.

This is not the ordinary determinant which gives the periods of vibration of a connected system of masses. The present determinant involves $1/\Omega^2$ instead of Ω^2 as in the ordinary one, from which the same result may be obtained. The present form, however, is more convenient for the numerical computation of whirling speeds.

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* Rayleigh, 'Collected Papers,' vol. 5, p. 170.

Expanding the determinant in powers of Φ we obtain

$$\Phi^n - \Phi^{n-1} \times (a_1 + b_2 + c_3 + \ldots + h_n) + \ldots = 0.$$

Thus we find for the sum of the roots

$$\Phi_1 + \Phi_2 + \Phi_3 + \ldots + \Phi_n = a_1 + b_2 + c_3 + \ldots + h_n.$$

If $k = 6\mathrm{EI}l$, we have

$$\Phi_1 = \frac{k}{\Omega_1^2}, \qquad \Phi_2 = \frac{k}{\Omega_2^2}, \text{ etc.}$$

Next let us consider each mass acting alone. We find, either directly, or from the above general equations on equating each of the remaining masses to zero,

$$\phi u_1 = a_1 u_1, \qquad \phi u_2 = b_2 u_2, \text{ etc.}$$

Hence
$$\phi_1 = a_1, \qquad \phi_2 = b_2, \text{ etc.}$$

Also
$$\phi_1 = \frac{k}{\omega_1^2}, \qquad \phi_2 = \frac{k}{\omega_2^2}, \text{ etc.}$$

We thus see that

$$\frac{1}{\Omega_1^2} + \frac{1}{\Omega_2^2} + \ldots + \frac{1}{\Omega_n^2} = \frac{1}{k}(a_1 + b_2 + \ldots + h_n) = \frac{1}{\omega_1^2} + \frac{1}{\omega_2^2} + \ldots + \frac{1}{\omega_n^2},$$

i.e.
$$\Sigma \frac{1}{\Omega^2} = \Sigma \frac{1}{\omega^2}.$$

Thus the sum of the squares of the reciprocals of the several whirling speeds of a shaft carrying a number of loads is equal to the sum of the squares of the reciprocals of the whirling speed of each mass regarded as acting alone.

Now, $\Omega_1, \Omega_2, \Omega_3, \ldots, \Omega_n$, being the several whirling speeds of the loaded shaft, are an increasing series of numerical values. If Ω_2, Ω_3, etc., are considerably greater than Ω_1, we may to a first approximation neglect the sum of $1/\Omega_2^2$, $1/\Omega_3^2$, etc., compared with $1/\Omega_1^2$. We then have, approximately, provided we are justified in the omission of Ω_2, Ω_3, etc.,

$$\frac{1}{\Omega_1^2} = \frac{1}{\omega_1^2} + \frac{1}{\omega_2^2} + \ldots + \frac{1}{\omega_n^2}.$$

This is Dunkerley's empirical rule.

Its degree of approximation then clearly depends on the values of the second and higher whirling speeds of the loaded shaft relatively to the first. Necessarily, it gives too low a value for the first whirling speed.

To obtain a numerical idea of the degree of approximation, it will be convenient to consider the case of a uniformly loaded and uniform shaft, in which the equations giving the several whirling speeds under different conditions of support are well known.

Let q stand for the ratio $\dfrac{\Sigma(1/\Omega^2)}{(1/\Omega_1^2)}$.

A. Taking first the case of a shaft fixed in direction at one end and free at the other. Then* the natural frequencies must satisfy

$$\cos\mu\cosh\mu+1 = 0,$$

where in our application μ^2 is proportional to Ω.

Accordingly, the frequencies of the tones or the values of the several whirling speeds are proportional to 1, 6·27, 17·55, 34·39, etc.

Further,† the expansion of the equation in μ leads to

$$1-\frac{\mu^4}{12}+\ldots = 0,$$

so that $$\Sigma\,\frac{1}{\mu^4} = \frac{1}{12}.$$

But μ^4 is $(1{\cdot}875104)^4$ or 12·3624.

Hence $$q = \frac{\Sigma(1/\mu^4)}{(1/\mu_1^4)} = 1{\cdot}0302.$$

B. Consider next the case in which the shaft is simply supported at each end. Then‡ the natural frequencies must satisfy $\sin\mu = 0$.

Thus the frequencies of the tones or the values of the several whirling speeds are proportional to 1, 4, 9, 16, etc.

Also in this case the expansion of the equation in μ is

$$\mu\left(1-\frac{\mu^2}{6}+\frac{\mu^4}{120}-\ldots\right) = 0,$$

so that $$\Sigma\,\frac{1}{\mu^2} = \frac{1}{6},\quad \Sigma\,\frac{1}{\mu_1^2\mu_2^2} = \frac{1}{120};\quad \text{therefore } \Sigma\,\frac{1}{\mu^4} = \frac{1}{90}.$$

But $\mu_1 = \pi$.

Therefore $$q = \frac{\Sigma(1/\mu^4)}{(1/\mu_1^4)} = \frac{\pi^4}{90} = 1{\cdot}0823.$$

C. If the shaft be fixed in direction at one end and supported at the other, the equation for μ is§

$$\cot\mu = \coth\mu,$$

which on expansion gives $1-\dfrac{\mu^4}{210}+\ldots = 0$. The frequencies are proportional to 1, 3·24, 6·76, 11·56, etc.

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* See Rayleigh, 'Theory of Sound,' vol. 1, § 173, and Dunkerley, *ibid.*, § 9.

† Rayleigh, *ibid.*, § 175.

‡ Rayleigh, *ibid.*, §181 ; Dunkerley, *ibid.*, §10.

§ Rayleigh, *ibid.*, §§ 180, 174 ; Dunkerley, *ibid.*, §14.

Also $$\Sigma\frac{1}{\mu^4}=\frac{1}{210} \quad \text{and} \quad \mu_1 = 3{\cdot}9266,$$

so that $$\mu_1{}^4 = 237{\cdot}721.$$

Therefore $$q=\frac{\Sigma(1/\mu^4)}{(1/\mu_1{}^4)} = 1{\cdot}132.$$

D. Shaft fixed in direction at each end.

The equation for μ is*

$$\cos\mu\cosh\mu - 1 = 0,$$

which reduces to $$1-\frac{\mu^4}{420}+\ldots = 0.$$

The frequencies are proportional to 1, 2·76, 5·40, 8·93, etc.

Hence $$\Sigma\frac{1}{\mu^4}=\frac{1}{420}.$$

But $$\mu_1{}^4 = (4{\cdot}73004)^{4}\dagger = 500{\cdot}564.$$

Therefore $$q=\frac{\Sigma(1/\mu^4)}{(1/\mu_1{}^4)} = 1{\cdot}192.$$

E. Finally take the case of a shaft simply supported in three bearings, the spans being of equal length.

Then‡ μ must satisfy either

$$\sin\mu = 0, \qquad \text{or} \qquad \cot\mu = \coth\mu.$$

The frequencies are proportional to 1, 1·56, 4, 5·06, etc. In the former case $\Sigma(1/\mu^4) = 1/90$, as in (B) above; for the latter $\Sigma(1/\mu^4) = 1/210$, as in (C). For both sets of frequencies, therefore, we have $\Sigma(1/\mu^4) = 1/63$.

But $\mu_1 = \pi$.

Hence $$q=\frac{\Sigma(1/\mu^4)}{(1/\mu_1{}^4)} = \frac{\pi^4}{63} = 1{\cdot}5462.$$

Summing up these results, we see that the values of Ω_1 obtained by omitting the higher whirling speeds are thus too low, in the following ratios:—

0·985, 0·961, 0·940, 0·916, 0·804.

Thus in some of these cases the agreement is good; in the last it is quite insufficient, and this is not necessarily the worst case that may arise.

These results may also be obtained otherwise, since we know for these several systems of support the whirling speed of a massless shaft carrying an isolated mass at any point along its length. A uniformly loaded shaft may be considered made up of an infinite number of elements of mass spread along the shaft, and we can, therefore, by integration obtain the value of $\Sigma(1/\omega^2)$, and compare it with the exact value of $1/\Omega_1^2$ in each case.

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* Rayleigh, *ibid.*, §172; Dunkerley, *ibid.*, §17.

† Rayleigh, *ibid.*, §174.

‡ Dunkerley, *ibid.*, §15.

To obtain a better approximation in cases where Ω_2 is not much greater than Ω_1, and where in consequence the Dunkerley rule is not sufficiently exact, the preceding determinant is to be expanded to a further power of Φ thus

$$\Phi^n - \Phi^{n-1}(a_1+b_2+c_3+\ldots+h_n)$$
$$+\Phi^{n-2}\{(a_1b_2)+(a_1c_3)+\ldots+(b_2c_3)+\ldots+(g_{n-1}h_n)\}-\ldots = 0,$$

where (a_1b_2) stands for the determinant $\begin{vmatrix} a_1, & a_2 \\ b_1, & b_2 \end{vmatrix}$.

As before, the sum of the roots $\Phi_1+\Phi_2+\ldots+\Phi_n$ is equal to $a_1+b_2+\ldots+h_n$. Also the sum of the products of the roots taken two at a time

$$\Phi_1\Phi_2+\Phi_1\Phi_3+\ldots+\Phi_2\Phi_3+\ldots+\Phi_{n-1}\Phi_n$$
$$=(a_1b_2)+(a_1c_3)+\ldots+(b_2c_3)+\ldots+(g_{n-1}h_n).$$

But $\Phi_n = \dfrac{k}{\Omega_n^2}$, and $h_n = \dfrac{k}{\omega_n^2}$ as before.

Also (a_1b_2) is the product of the first and second whirling speeds of m_1 and m_2 acting together, but with the other masses omitted; and so on.

We thus find that

$$\Sigma\frac{1}{\Omega^2} = \Sigma\frac{1}{\omega^2} \quad \text{and} \quad \Sigma\frac{1}{\Omega_r^2\Omega_s^2} = \Sigma\frac{1}{\omega'^2\omega''^2},$$

where ω', ω'' are the first and second whirling speeds when the masses are taken as acting two at a time.

If now we may neglect Ω_3 and the higher whirling speeds we have

$$\frac{1}{\Omega_1^2}+\frac{1}{\Omega_2^2} = \Sigma\frac{1}{\omega^2},$$
$$\frac{1}{\Omega_1^2\Omega_2^2} = \Sigma\frac{1}{\omega'^2\omega''^2};$$

and from these two equations a more exact value of Ω_1 may be deduced, together with an approximate value of Ω_2.

In this case it becomes necessary to determine the whirling speeds for each of the masses regarded as acting alone, and also the first and second whirling speeds for the masses acting two at a time in all possible pairs.

Dunkerley's empirical method may thus be extended to the following rule :—

Find the whirling speed (ω) for each load acting separately; and the first and second whirling speeds (ω', ω'') for each combination of two loads acting without any of the others. Then if Ω_1 and Ω_2 are the first and second whirling speeds for the whole system, they are given by

$$\frac{1}{\Omega_1^2}+\frac{1}{\Omega_2^2} \doteqdot \Sigma \frac{1}{\omega^2},$$

$$\frac{1}{\Omega_1^2\Omega_2^2} \doteqdot \Sigma \frac{1}{\omega'^2\omega''^2}.$$

If we expand the determinantal equation to further powers of Φ we will find in the same manner that the sum of the squares of the reciprocals of the products of the n whirling speeds taken r at a time is equal to the sum of the squares of the reciprocals of the products of the r speeds for each set of r masses, the other $n-r$ masses being supposed reduced to zero.

Further, as a similar determinantal equation will obviously be obtained in any general case in which linear displacement equations of the preceding type apply, we may express with greater generality the results arrived at.

Thus the sum of the squares of the reciprocals of the products of the n speeds of vibration taken r at a time of n masses connected elastically, is equal to the sum of the squares of the reciprocals of the products of the r speeds of vibration for each set of r masses, the other $n-r$ masses being supposed to be obliterated.

The particular case when $r = 1$ is perhaps the most interesting. The sum of the squares of the reciprocals of the n speeds of vibration of the system of n masses is equal to the sum of the squares of the reciprocals of the speeds of vibration of each of the n masses regarded as acting alone.

Further, if the n masses be divided into any number of groups the sum of the squares of the reciprocals of all the speeds of vibration of each of the groups, taken for all the groups, is equal to the sum of the squares of the reciprocals of the n speeds of vibration for all the n masses acting together. The theorem admits of other generalisations.

In general, then, the n whirling speeds are the roots of the determinantal equation obtained above. In any particular application numerical values may be substituted for the masses, their positions, and the elastic constants of the shaft. The roots of the resulting numerical determinantal equation may be evaluated by successive approximation. Usually not more than the first two or possibly three speeds are required.

This determinantal method, although tedious when the number of isolated masses is great, is straightforward and exact. It may be made to take account of variations of shaft diameter, and also of the moments of inertia of the loads, and of different bearing conditions. It is quite general and makes no assumption of an approximate displacement curve of gravitational or other type.

In cases where with sufficient accuracy whirling speeds higher than the second may be neglected we have, in the foregoing notation,

$$\frac{1}{\Omega_1^2}+\frac{1}{\Omega_2^2} \doteqdot a_1+b_2+c_3+\ldots+h_n,$$

and

$$\frac{1}{\Omega_1^2\Omega_2^2} \doteqdot (a_1b_2)+(a_1c_3)+\ldots+(b_2c_3)+\ldots+(g_{n-1}h_n);$$

i.e. the sum of all the second order determinants about the principal diagonal.

Ω_1 and Ω_2 may thence be determined.

On the Spectrum of Cadmium in the Inactive Gases.

By J. N. Collie, F.R.S., and H. E. Watson, D.Sc.

(Received June 21, 1918.)

During the last few years some attention has been paid to the spectrum of gases in the neighbourhood of the cathode discharge, and, therefore, it is thought to be worth while putting on record some experiments that were made about ten years ago, but which were not considered at the time to be of sufficient importance to be published, as they led to no definite conclusion. One of the authors* had noticed that, in a vacuum tube containing helium and mercury vapour, the orange mercury line 6152 appeared very brightly, although it is not usually seen at all in a tube containing a diatomic gas. Continuing this line of research, other metals were experimented on, with those metals as electrodes. It was at once noted that in many cases, at comparatively high pressures and low current densities, lines in the spectrum of the metals composing the electrodes were visible in the neighbourhood of the cathode, and often in other parts of the tube as well.

The following experiments were made, with the view of studying more thoroughly this effect. Cadmium was the metal chosen, on account of the ease with which the metallic spectrum was produced, and also because the cadmium bright lines extend over a considerable portion of the visible spectrum. Other metals that behave in a similar manner are zinc, magnesium, lead, bismuth. The gases experimented with were helium, neon, argon, and xenon.

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* 'Roy. Soc. Proc.,' vol. 71, p. 25.